UDC 539.3

ON SOME DIRECT METHODS AND THE EXISTENCE OF SOLUTION IN THE NONLINEAR THEORY OF ELASTIC NONSHALLOW SHELLS OF REVOLUTION

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The existence of a generalized solution is proved by the methods of [1] and the convergence of the approximate Ritz and Bubnov-Galerkin methods in the problem of the equilibrium of an elastic nonshallow shell whose middle surface being part of a surface of revolution is given a foundation for an arbitrary load and a rigidly supported edge. In the particular case of axisymmetric strain of a shell of revolution, an important topological characteristic of the problem, the rotation of the vector field, is calculated.

1. Fundamental relationships. The following version of the relationships of nonlinear theory of nonshallow shells is considered, which can be obtained from the relationships for mean bending:

$$T_{ij}(\varepsilon_{kl}) = E_{ijkl}\varepsilon_{kl}, \qquad M_{ij} = D_{ijkl}\varkappa_{kl} \tag{1.1}$$
$$\varepsilon_{ij} = e_{ij} + {}^1/_2\psi_i\psi_j, \qquad \psi_1 = A_1^{-1}w_{\alpha_1} - R_1^{-1}u_1$$
$$e_{11} = A_1^{-1}u_{1\alpha_1} + A_{1\alpha_2}(A_1A_2)^{-1}u_2 + R_1^{-1}w$$
$$2e_{12} = A_1A_2^{-1}(A_1^{-1}u_1)_{\alpha_2} + A_2A_1^{-1}(A_2^{-1}u_2)_{\alpha_1}$$
$$\varkappa_{11} = -A_1^{-1}\psi_{1\alpha_1} - A_{1\alpha_2}(A_1A_2)^{-1}\psi_2$$
$$2\varkappa_{12} = -A_1A_2^{-1}(A_1^{-1}\psi_1)_{\alpha_2} - A_2A_1^{-1}(A_2^{-1}\psi_2)_{\alpha_1}$$
$$E_{ijkl} = E_{klij}, \qquad D_{ijkl} = {}^1/_3h^2E_{ijkl} \quad (1 \rightleftarrows 2)$$

Here T_{ij} are the tangential stress resultants, ε_{ij} are the tension and shear strains, M_{ij} are the bending moments, $\varkappa_{ij}$ is the change in curvature R_k^{-1} of the shell middle surface S^* ; ψ_i are the angles of rotation of the coordinate lines α_i; A_i^2, $2\ C = 0$ are coefficients of the first quadratic form of the surface S^*; u_1, u_2, w are the displacements of points of the shell middle surface S^*; the subscript α_i denotes differentiation with respect to the coordinate α_i; $2h$ is the shell thickness, and E_{ijkl}, D_{ijkl} are elastic shell characteristics.

This version of the theory of nonshallow shells has already been examined in [2, 3], however, an erroneous proof of the fundamental a priori estimate of the solution of the problem was presented therein. The proof of the existence of a solution is carried out herein by a method analogous to that in [1].

Let the following conditions be satisfied:

1) The shell middle surface S^* is part of a surface of revolution, and the homeomorphic mapping of its meridian $\alpha_1 = \text{const}$ onto some segment $[a, b]$ is carried out by the function $r(\alpha_2) \in C^{(3)}(a, b)$;

2) The domain Ω occupied by the shell planform is the finite sum of bounded star-shaped domains, if the shell is closed (see Fig. 1), then the domain included between the lines $\alpha_1 = 0$ and $\alpha_1 = 2\pi$ is taken as Ω ;

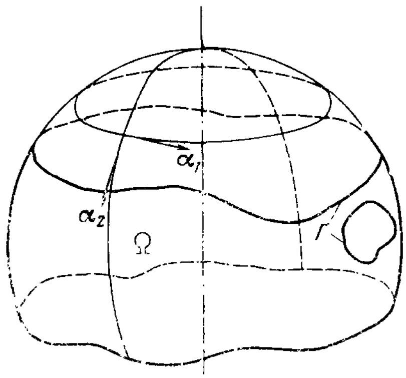

Fig. 1

3) The boundary Γ of the domain Ω consists of a finite number of closed contours of the Liapunov class $Л_1\,(m,\,0)$ (*) ;

4) The inequalities

$$0 < m_1 \leqslant A_i, \qquad h,\ |\,R_i\,| \leqslant m_2$$

are valid everywhere in the domain Ω (here and henceforth $m_k > 0$ are some positive constants);

5) E_{ijkl} are piecewise-continuous functions in Ω , where the inequality

$$m_3 \varepsilon_{ij} \varepsilon_{ij} \leqslant E_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \leqslant m_4 \varepsilon_{ij} \varepsilon_{ij}$$

is satisfied in Ω for all symmetric tensors ε_{ij} .

Condition (4) permits elimination of components of the displacement vector u_1, u_2 from all the relationships (1.1) by using the relationship

$$u_i = R_i A_i^{-1} w_{\alpha_i} - R_i \psi_i, \qquad i = 1, 2$$

This substitution is considered to be carried out everywhere without any additional stipulations.

The Lagrange principle determines the shell equilibrium equation

$$\int_\Omega \{T_{ij}(\varepsilon_{kl})\,\delta\varepsilon_{ij} + M_{ij}\delta\varkappa_{ij}\}\,A_1 A_2\,d\alpha_1\,d\alpha_2 = \tag{1.2}$$

$$\int_\Omega \{F_1 (R_1 A_1^{-1} \delta w_{\alpha_1} - R_1 \delta\psi_1) + F_2 (R_2 A_2^{-1} \delta w_{\alpha_2} - R_2 \delta\psi_2) + F_3 \delta w\}\,A_1 A_2\,d\alpha_1\,d\alpha_2$$

if the shell edge is rigidly fixed [clamped], i.e.

$$\psi_i\,|_\Gamma = 0, \qquad w\,|_\Gamma = 0, \qquad w_{\alpha_i}|_\Gamma = 0, \quad i = 1, 2 \tag{1.3}$$

Here F_i are components of the external load vector; the variational sign δ means that the "possible" displacement $\delta\omega$ $(\delta\psi_1, \delta\psi_2, \delta w)$, must be substituted in the corresponding expression in place of the vector function ω (ψ_1, ψ_2, w) , where

$$\delta\varepsilon_{ij} = \delta e_{ij} + {}^1/_2\,(\psi_i \delta\psi_j + \psi_j \delta\psi_i)$$

A system of three differential equations in the vector function ω (ψ_1, ψ_2, w) can be obtained from (1.2) by a method standard for calculus of variations. This system is equivalent to the ordinary system of nonshallow shell equilibrium equations in the displacements u_1, u_2, w

*) Editorial Note. Cyrillic symbol Л derives from Liapunov (Ляпунов).

The following scalar product

$$(\omega \cdot \delta\omega)_{\mathrm{H}} = \int_{\Omega} \{T_{ij}(e_{kl})\,\delta e_{ij} + M_{ij}\delta\varkappa_{ij}\}\, A_1 A_2\, d\alpha_1\, d\alpha_2 \qquad (1.4)$$

is introduced.

Definition 1.1. The space H is the closure of the set C of vector functions $\omega(\psi_1, \psi_2, w) \in C^{(1)}(\Omega) \times C^{(1)}(\Omega) \times C^{(2)}(\Omega)$ satisfying the boundary conditions (1.3), in the form corresponding to the scalar product (1.4). In the case of a closed shell (see Condition (2)) still 2π-periodicity in the variable α_1 must be required of these functions.

As in [4], the following lemma can be proved.

Lemma 1.1. Let Conditions (1) — (5) be satisfied. In this case the space H is a subspace of

$$\mathrm{W} = W_2^1(\Omega) \times W_2^1(\Omega) \times W_2^2(\Omega)$$

where the inequality

$$0 < m_5 \leqslant \|\omega\|_{\mathrm{H}} \|\omega\|_{\mathrm{W}}^{-1} \leqslant m_6$$

is valid for an arbitrary element $\omega \in \mathrm{H}$, with the constants m_5, m_6 not dependent on the selection of $\omega \in \mathrm{H}$. Moreover, the space formed by the closure of the subset of vector functions $\mathrm{a} \in \mathrm{C}$ of the form $\mathrm{a} = (\psi_1, \psi_2, 0)$ in the norm corresponding to the scalar product in H_1

$$(\mathrm{a} \cdot \delta \mathrm{a})_{\mathrm{H}_1} = \int_{\Omega} M_{ij}\delta\varkappa_{ij} A_1 A_2\, d\alpha_1\, d\alpha_2$$

is a subspace $\mathrm{W}_1 = W_2^1(\Omega) \times W_2^1(\Omega)$, where the norms of H_1 and W_1 are equivalent in the space H_1.

Lemma 1.1 shows that the corresponding Sobolev's imbedding theorems [5] are valid for the space H, H_1.

2. Formulation of the problem. As in [6], the concept of a generalized solution is introduced.

Definition 2.1. The generalized solution of the problem of equilibrium of an elastic nonshallow shell with rigidly fixed edge is the vector function $\omega(\psi_1, \psi_2, w) \in \mathrm{H}$ such that for an arbitrary vector function $\delta\omega \in \mathrm{H}$ the integral relationship (1.2) is satisfied.

By using the Hölder inequality and Lemma 1.1 it can be shown that all the terms of (1.2) have meaning for such a definition of the generalized solution, and moreover, each is a continuous linear functional in the variable $\delta\omega$ in the space H if there is compliance with the condition

6) $F_1, \quad F_2 \in L_p(\Omega), \qquad p > 1, \qquad F_3 \in L(\Omega)$

On the basis of the Riesz theorem about the representation of a continuous linear functional in Hilbert space, (1.2) can be written as an operator equation in the space H

$$\omega = \mathrm{G}\omega$$

Considering the variation of the energy functional J, it can be shown that

$$\mathrm{I} - \mathrm{G} = \mathrm{grad}_{\mathrm{H}} J$$

$$J = \frac{1}{2}\|\omega\|_{\mathrm{H}}^2 + J_2 = \frac{1}{2}\int_{\Omega} \{T_{ij}(\varepsilon_{kl})\,\varepsilon_{ij} + M_{ij}\varkappa_{ij}\}\, A_1 A_2\, d\alpha_1\, d\alpha_2 -$$

$$\int_{\Omega} \{F_1(R_1 A_1^{-1} w_{\alpha_1} - R_1\psi_1) + F_2(R_2 A_2^{-1} w_{\alpha_2} - R_2\psi_2) + F_3 w\}\, A_1 A_2\, d\alpha_1\, d\alpha_2$$

It follows from the explicit form of the functional J_2 and Lemma 1.1 that J_2 is weakly continuous, and in turn, this implies the complete continuity of the operator $\mathbf{G}$ by the E. S. Tsitlanadze theorem.

The existence of a generalized solution will have been proved if the existence of critical points of the functional J is shown (see [1]).

Lemma 2.1. Let Conditions (1) – (5) be satisfied, and let there be a sequence ω_n weakly convergent to the element ω_0 in the space $\mathbf{H}$ such that $J_4(\omega_n) \to 0$. In this case $\omega_0 = 0$. Here $J_4(\omega)$ is a part of the functional J homogeneous in R to the fourth power for the mapping of the unit sphere $S\{\omega^* \in S: \|\omega\|_{\mathbf{H}} = 1\}$ on the "ellipsoid" $C(R)$ defined by the relations

$$w = R^2 w^*, \; \psi_i = R\psi_i^*, \quad i = 1, 2, \quad \omega^* \in S, \; \omega \in C(R)$$

Without limiting the generality, it can be considered that $\omega_n \in \mathbf{C}$.

There results from the form of the functional $J_4(\omega)$ and Condition (5) that the quantities

$$\gamma_{1n} = \frac{(R_1 A_1^{-1} w_{n\alpha_1})_{\alpha_1}}{A_1} + \frac{A_{1\alpha_2} R_2 w_{n\alpha_2}}{A_1 A_2^2} + \frac{w_n}{R_1} + \frac{1}{2}\psi_{1n}^2 \quad (1 \rightleftarrows 2) \quad (n \to \infty) \tag{2.1}$$

should tend to zero in $L_2(\Omega)$ and therefore also in $L(\Omega)$. Multiplying the second of the relations (2.1) by $A_1 A_2 R_1^{-1}$, integrating over the domain Ω and relying on the Gauss-Codazzi relationships

$$(A_2^{-1} A_{1\alpha_2})_{\alpha_2} + (A_1^{-1} A_{2\alpha_1})_{\alpha_1} + A_1 A_2 R_1^{-1} R_2^{-1} = 0$$

$$(A_1 R_1^{-1})_{\alpha_2} = A_{1\alpha_2} R_2^{-1} \quad (1 \rightleftarrows 2)$$

we can obtain by elementary manipulation

$$\frac{1}{4}\int_\Omega \psi_{2n}^2 R_1^{-1} A_1 A_2 \, d\alpha_1 d\alpha_2 = \int_\Omega \left\{\left(\frac{A_{2\alpha_1}}{A_1}\right)_{\alpha_1} w_n - \frac{1}{2}\gamma_{2n} R_1^{-1} A_1 A_2\right\} d\alpha_1 d\alpha_2$$

Hence it follows (since $A_{2\alpha_1} \equiv 0$) that

$$\int_\Omega \psi_{2n}^2 \, d\alpha_1 d\alpha_2 \to 0$$

i.e. $\psi_{20} = 0$. There results from Lemma 1.1 and the second of the equalities (2.1) that $\theta_n \to 0$ in the space $L_2(\Omega)$

$$\theta_n = A_2^{-1}(R_2 w_{n\alpha_2} A_2^{-1})_{\alpha_2} + w_n R_2^{-1} \tag{2.2}$$

Just as had been done in proving the imbedding theorems in [4], it can be shown that strong convergence of the sequences w_n, $w_{n\alpha_2}$, $w_{n\alpha_2\alpha_2}$ to zero in the space $L_2(\Omega)$ follows from the relationship (2.2), i.e. $w_0 = 0$. Finally, $\psi_{10} = 0$ results from the first of the relationships (2.1), which terminates the proof.

Lemma 2.2. If Conditions (1) – (6) are satisfied, then for sufficiently large $R > 0$ the following estimate

$$J \geqslant m_7 R^2, \qquad m_7 > 0 \tag{2.3}$$

is valid on the ellipsoids $C(R)$. The proof of the inequalities (2.3) is carried out as in [4]. The original S of the ellipsoid $C(R)$ is separated into three parts, S_1, S_2 S_3. For a sufficiently small, but completely definite $\varepsilon > 0$, the functional $J_4(\omega^*) \geqslant m_8$, $m_8 > 0$, on $S_1\{\omega^*(\mathbf{a}^*, w^*) \in S_1: \|\mathbf{a}^*\|_{\mathbf{H}_1}^2 \leqslant \varepsilon\}$ and the estimate (2.3) is satisfied on this part of $C(R)$ for the mapping of S_1 on $C(R)$ since the remaining terms in the functional J have powers of R not greater than the third.

The set S_2
$$\frac{1}{2}\|\omega^*\|_{\mathbf{H}}^2 - \int_\Omega \{F_1A_2R_1w_{\alpha_1}^* + F_2A_1R_2w_{\alpha_2}^* + F_3A_1A_2w^*\}\, d\gamma_1 d\gamma_2 \geqslant \frac{1}{4}\varepsilon$$
is separated out of the remaining part of the sphere $S \setminus S_1$.

It follows from the form of the functional J that the estimate (2.3) is satisfied with the constant $m_7 = {}^1/_5\,\varepsilon$ on the mapping S_2. The set $S_3 = S \setminus (S_1 \cup S_2)$ contains no weak zero. There results from Lemma 2.1 that the functional $J_4(\omega^*)$ is strictly positive on S_3, from which the estimate (2.3) on the mapping S_3 in $C(R)$ follows, as above.

For $R > 0$ the ellipsoid $C(R)$ is the boundary of a connected convex set which is star-shaped relative to zero in the space $\mathbf{H}$. There results from Lemma 2.2 that there is a minimum point within some ellipsoid $C(R)$ with a sufficiently large diameter R.

Indeed, let d_0 be the exact lower bound of the function J in the space $\mathbf{H}$. There results from Lemma 2.2 that $d_0 > -\infty$. Every minimizing functional J of the sequence ω_n evidently lies in the set M defined by the inequality $J \leqslant d_0 + m_9$. Because of the estimate (2.3) this set lies in some ellipsoid which is a bounded set in $\mathbf{H}$, and therefore, the set $\{\omega_n\}$ is weakly compact. Since the functional J_2 is weakly continuous, then by repeating the discussion in [1] word for word, it can be obtained that the set $\{\omega_n\}$ is strongly compact and each weak limit ω_0 of the sequence ω_n is simultaneously the strong limit. Therefore
$$J(\omega_0) = \lim J(\omega_n) = d_0$$
this terminates the proof of the existence of critical points of the functional J.

The following theorem is therefore valid.

Theorem 2.1. Let Conditions (1)–(6) mentioned above be satisfied. In this case, there exists at least one generalized solution of the problem in the sense indicated in Definition 2.1.

3. Convergence of the Bubnov-Galerkin method. The foundation of the Bubnov-Galerkin method in nonshallow shell theory is carried out by the same scheme as for shallow shells [7].

Let χ_l be a complete orthonormalized system of vector functions in the space $\mathbf{H}$. The generalized solution of the problem is sought approximately by the Ritz method in the form
$$\omega_n = \sum_{l=1}^{n} q_{nl}\chi_l \tag{3.1}$$
as the minimal value of the functional J in an n-dimensional manifold M_n extended over the vectors χ_l, $l = 1, \cdots, n$ from the following system of algebraic equations:
$$\frac{\partial}{\partial q_{nl}} J(\omega_n) = 0, \qquad l = 1, \ldots, n \tag{3.2}$$

It can be shown because of the estimate (2.3) that the inequality
$$J(0) \leqslant J(\omega_n), \qquad \omega_n \in C_n(R)$$
is valid for sufficiently large values of $R > 0$ in ellipsoids $C_n(R)$ in the n-dimensional space of coefficients q_{nl} obtained from the ellipsoids $C(R)$ by extraction of points belonging to the set M_n. Therefore the functional J considered in the set M_n

takes on the minimal value within an ellipsoid $C_n\ (R)$. There hence results that the system (3.2) has at least one real solution within the ellipsoid $C_n(R)$ for sufficiently large $R > 0$. This same solution also lies within the ellipsoid $C\ (R)$ of the space $\mathbf{H}$ independently of n, i.e. the sequence of approximate Ritz solutions is weakly compact in the space $\mathbf{H}$. Strong compactness of the sequence of approximate Ritz solutions can be shown analogously [1].

The system of equations of the Bubnov-Galerkin method are constructed in a n-th approximation as follows: in place of the vector function ω in (1.2),(3.1) is substituted and the n vector functions of the basis χ_l, $l = 1, \cdots, n$ are substituted successively in place of $\delta\omega$. The following theorem results from the explicite form of the functional J and its properties, analogously to [1].

Theorem 3.1. Let all the conditions of Theorem 2.1 be satisfied and let χ_l be a complete orthonormal system of vector functions in the space $\mathbf{H}$. If the generalized solution of the problem is sought approximately by the Ritz or Bubnov-Galerkin methods, then the following assertions are valid:

1) The same system of algebraic equations, having at least one real solution, is always obtained for the coefficients q_{nl} when both methods are used;

2) The set of approximate solutions ω_n included in a sphere of the space $\mathbf{H}$ of sufficiently large radius, is infinite, strongly compact, and contains a sequence minimizing J;

3) Each limit point of the set of approximate solutions ω_n is the generalized solution of the problem in the sense of Definition 2.1.

Note. All the theorems obtained above remain valid if only $A_{2\alpha_1} \equiv 0$ is assumed instead of the condition that S^* is part of a surface of revolution.

4. Axisymmetric problem. The axisymmetric case of shell deformation is presented here for two reasons: firstly, the existence of an axisymmetric solution in the case of deformation of a shell of revolution by an axisymmetric load does not follow from Theorem 2.1, and secondly, the topological characteristic of the problem, the rotation of a vector field, is calculated in this case.

The middle surface S^* of an axisymmetrically deformable isotropic homogeneous elastic shell is a part of a surface of revolution enclosed between the two parallels $\alpha_2 = a$ and $\alpha_2 = b$. The fundamental relationships (1.1) become in this case

$$\begin{aligned}
&T_{11}\ (\varepsilon_{kl}) = E_1\ (\varepsilon_{11} + \nu\varepsilon_{22}), \qquad T_{12} = 0 \qquad\qquad (4.1)\\
&M_{11} = D\ (\varkappa_{11} + \nu\varkappa_{22}), \qquad M_{12} = 0 \qquad (1 \rightleftarrows 2)\\
&\varepsilon_{ij} = e_{ij} + {}^1/_2\ \psi_i\psi_j\\
&\varepsilon_{11} = A_{1\beta}\ (A_1A_2)^{-1}\ (R_2A_2^{-1}w_\beta - R_2\psi) + R_1^{-1}w\\
&\varepsilon_{22} = A_2^{-1}(R_2A_2^{-1}\ w_\beta - R_2\psi)_\beta + wR_2^{-1} + {}^1/_2\psi^2\\
&\varepsilon_{12} = \varepsilon_{21} = \psi_1 = \varkappa_{12} = \varkappa_{21} = 0, \qquad\qquad \psi_2 = \psi\\
&\varkappa_{11} = -A_{1\beta}\ (A_1A_2)^{-1}\ \psi, \qquad \varkappa_{22} = -A_2^{-1}\psi_\beta\\
&E_1 = 2hE\ (1-\nu^2)^{-1}, \qquad D = {}^2/_3h^3\ E\ (1-\nu^2)^{-1}, \qquad 0 < \nu < {}^1/_2.
\end{aligned}$$

Here the coordinate α_2 is renamed β, E is the Young's modulus, and ν is the Poisson's ratio. All the functions in (4.1) depend only on the coordinate β.

Equation (1.2) can be rewritten in this case as two equations in the functions ψ and w

$$\int_a^b \{M_{ij}\delta\varkappa_{ij}A_1A_2 - T_{11}A_{1\beta}R_2\delta\psi - T_{22}A_1(R_2\delta\psi)_\beta + T_{22}A_1A_2\psi\delta\psi\}\,d\beta = -\int_a^b F_2R_2\delta\psi A_1A_2d\beta \tag{4.2}$$

$$\int_a^b T_{ij}(\varepsilon_{kl})\,\delta\gamma_{ij}A_1A_2d\beta = \int_a^b \{F_2R_2A_1\delta w_\beta + F_3A_1A_2\delta w\}\,d\beta \tag{4.3}$$

Here

$$\gamma_{11} = A_{1\beta}(A_1A_2^2)^{-1}R_2w_\beta + wR_1^{-1}$$

$$\gamma_{22} = A_2^{-1}(R_2w_\beta A_2^{-1}) + wR_2^{-1}$$

The boundary conditions (1.3) become

$$\psi(a) = \psi(b) = w(a) = w(b) = w'(a) = w'(b) = 0 \tag{4.4}$$

The conditions corresponding to Conditions (1) – (6) are written as follows:

1) The shell middle surface S^* is a part of the sutface of revolution included between the parallels $\beta = a$ and $\beta = b$; the homeomorphic mapping of its meridian on the segment $[a, b]$ is made by the function $r(\beta) \in C^{(3)}(a, b)$;

2) The following inequalities are satisfied:

$$0 < m_{10} \leqslant A_i, \quad h, |R_i|, E \leqslant m_{11}, \ 0 < \nu < 1/2$$

3) The external stress resultants satisfy the requirements

$$F_2 \in \overset{\circ}{W}{}_2^{-1}(a, b), \quad F_3 \in \overset{\circ}{W}{}_2^{-2}(a, b)$$

Definition 4.1. Closure of the set of functions $\psi \in C^{(1)}(a, b)$ satisfying the boundary conditions (4.4) in the norm corresponding to the scalar product

$$(\psi\cdot\delta\psi)_B = \int_a^b M_{ij}\delta\varkappa_{ij}A_1A_2d\beta \tag{4.5}$$

is called the space $\mathbf{B}$.

Definition 4.2. Closure of the set of functions $w \in C^{(2)}(a, b)$ satisfying the boundary conditions (4.4) in the norm corresponding to the scalar product

$$(w\cdot\delta w)_S = \int_a^b T_{ij}(\gamma_{kl})\,\delta\gamma_{ij}A_1A_2d\beta \tag{4.6}$$

is called the space S.

As in [4], the proof is carried out by the following lemma.

Lemma 4.1. Let Conditions (1) and (2) (of Sect. 4) be satisfied. In this case, the spaces $\mathbf{B}$ and S agree, respectively, with the spaces $W_2^{\circ 1}(a, b)$ and $W_2^{\circ 2}(a, b)$, where the norms (4.5) and (4.6) are equivalent to the ordinary norms of the spaces $W_2^{\circ 1}(a, b)$ and $W_2^{\circ 2}(a, b)$, respectively.

Definition 4.3. The pair of functions $\psi \in \mathbf{B}$, $w \in S$ satisfying the equations (4.2), (4.3) for any pair of functions $\delta\psi \in \mathbf{B}$, $\delta w \in S$ is called the generalized axisymmetric solution of the problem of equilibrium of an elastic shell of revolution with rigidly fixed edge subjected to an axisymmetric load.

All the members of (4.2) and (4.3) have meaning in such a definition of the generalized solution if Conditions (1) – (3) (of Sect. 4) are satisfied.

Using Lemma 4.1, as well as the Riesz theorem about the representation of a continuous linear functional in Hilbert space, (4.3) can be written as an operator equation in the space S

$$(w\cdot\delta w)_S = \sum_{i=0}^{2} (\mathbf{K}_i\psi\cdot\delta w)_S \tag{4.7}$$

Here $\mathbf{K}_i$ are continuous homogeneous operators in the variable ψ from the space $\mathbf{B}$ in S; i is the degree of homogeneity of the operator $\mathbf{K}_i$. It can be shown that $\mathbf{K}_2$ is a completely continuous operator.

Substituting the expression for the function w from (4.7) into (4.2) and still using the Riesz theorem, we can arrive at the operator equation in the space $\mathbf{B}$

$$\psi = \mathbf{G}_1\psi \tag{4.8}$$

whose solution is equivalent to finding the generalized solution.

The operator $\mathbf{G}_1$ is the sum of two operators: $\mathbf{G}_1 = \mathbf{L}_1 + \mathbf{G}_2$, where $\mathbf{G}_2$ is a completely continuous operator (the proof is carried out just as in [6]), and the linear continuous operator $\mathbf{L}_1$ is given by the relationship

$$(\mathbf{L}_1\psi\cdot\delta\psi)_\mathbf{B} = -\int_a^b T_{ij}(e_{kl}^{(1)})\,\delta\varphi_{ij}A_1A_2\,d\beta, \qquad \varphi_{ij} = e_{ij} - \gamma_{ij}$$

$e_{kl}^{(1)}$ is obtained from e_{kl} by the substitution $w = \mathbf{K}_1\psi$.

The following inequality

$$1 \leqslant \| \mathbf{I} - t\mathbf{L}_1 \| \leqslant m_{12}, \qquad \text{if} \quad 0 \leqslant t \leqslant 1 \tag{4.9}$$

results from the form of the operator $\mathbf{L}_1$.

In order to use the Leray-Schauder principle [8] on the fixed points of operators, two lemmas are proved.

Lemma 4.2. If the sequence $\psi_n \to \psi_0$ converges weakly in the space $\mathbf{B}$, the sequence $w_n \to w_0$ converges weakly in the space S and $J_4^*(\psi_n, w_n) \to 0$, then $\psi_0 = 0$. Here, the functional $J_4^*(\psi, w)$ is obtained from the functional $J_4(\omega)$ (Sect. 1) in an obvious manner.

The proof is analogous to the proof of Lemma 2.1.

Lemma 4.3. If Conditions (1) – (3) (Sect. 4) are satisfied, then the estimate

$$\Phi(\psi,\ t) = (\psi - t\mathbf{G}_1\psi\cdot\psi)_\mathbf{B} \geqslant m_{13}R^2 \tag{4.10}$$
$$\psi \in S(R), \quad 0 \leqslant t \leqslant 1, \quad m_{13} > 0$$

is valid on the spheres $S(R)$, $\{\psi \in S(R): \|\psi\|_\mathbf{B} = R\}$, of sufficiently large radius R

Assuming

$$\delta w = \chi_0 = -\int_a^\beta \psi(\lambda)\,A_2(\lambda)\,d\lambda$$

in (4.3), and taking it into account, the expression $\Phi(\psi, t)$ can be reduced to

$$\Phi(\psi, t) = \|\psi\|_\mathbf{B}^2 + 2t\int_a^b \Big\{T_{ij}(\varepsilon_{kl})\,\varepsilon_{ij} -$$

$$\frac{1}{2}\left[T_{11}(\varepsilon_{kl}) R_1^{-1} + T_{22}(\varepsilon_{kl}) R_2^{-1}\right]\chi_0\Big\} A_1 A_2 d\beta -$$

$$2t\int_a^b \left\{F_2 R_2 A_2^{-1} w_\beta + F_3 w - F_2 R_2 \psi + \frac{1}{2} F_3 \chi_0\right\} A_1 A_2 d\beta$$

The structure of the functional $\Phi(\psi, t)$ coincides with the structure of the corresponding functional $\Phi(w, t)$ in [6] and the estimate (4.10) is proved analogously to [6], taking account of Lemma 4.2.

The following inequality is evident

$$\Phi(\psi, t) \leqslant \| \mathbf{I} - t\mathbf{L}_1 \| \, \| \psi - t(\mathbf{I} - t\mathbf{L}_1)^{-1} \mathbf{G}_2 \psi \|_\mathrm{B} \, \| \psi \|_\mathrm{B} \qquad (4.11)$$

The following lemma results from inequalities (4.9) – (4.11).

Lemma 4.4. The estimate

$$\| \psi - t(\mathbf{I} - t\mathbf{L}_1)^{-1} \mathbf{G}_2 \psi \|_\mathrm{B} \geqslant m_{13} m_{12}^{-1} R, \quad 0 \leqslant t \leqslant 1 \qquad (4.12)$$

is valid on the spheres $\psi \in S(R)$ of sufficiently large radius R if the Conditions (1) – (3) (Sect. 4) are satisfied.

The operator $(\mathbf{I} - t\mathbf{L}_1)^{-1} \mathbf{G}_2$ is completely continuous for $0 \leqslant t \leqslant 1$ because of the estimate (4.9). It follows from the estimate (4.12) that the completely continuous vector field $\mathbf{I} - (\mathbf{I} - \mathbf{L}_1)^{-1}\mathbf{G}_2$ on spheres $S(R)$ of sufficiently large radius R is homotopic [8] to the unit field $\mathbf{I}$, from which the following theorem results.

Theorem 4.1. Let Conditions (1) – (3) (Sect. 4) be satisfied. In this case there exists at least one generalized axisymmetric solution, in the sense of Definition 4.3, of the problem of equilibrium of an elastic nonshallow shell of revolution with rigidly fixed edge subjected to axisymmetric loading.

All generalized solutions are bounded

$$\| \psi \|_\mathrm{B} \leqslant R, \qquad \| w \|_\mathrm{S} \leqslant m_{14}$$

where R is a sufficiently large parameter defined in Lemma 4.3, where the rotation of the completely continuous vector field $\mathbf{I} - (\mathbf{I} - \mathbf{L}_1)^{-1}\mathbf{G}_2$ equals plus one on the spheres $S(R_1)$, $R_1 > R$.

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THEORY OF A TWO-DIMENSIONAL POTENTIAL FIELD IN PIECEWISE-NONHOMOGENEOUS ANISOTROPIC REGULAR MEDIA

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We consider a potential field in piecewise-nonhomogeneous media having a regular structure. The basic structure consists of a doubly-periodic system of groups of arbitrary heterogeneous anisotropic inclusions. The heterogeneous inclusions present in each of these groups possess the same periodicity as the basic structure; they thus form a substructure. The problem of uniquely determining the field in this structure reduces to a determination of the solutions of a second order homogeneous elliptic equation in each of the component domains, the solutions being required to satisfy coupling conditions on the interfaces of the media and also some additional relationships. This boundary-value problem reduces to a system of regular integral equations, which we prove to be solvable. Questions arise in connection with the modelling of piecewise-nonhomogeneous anisotropic regular structures of a general type by means of homogeneous anisotropic media. As applications, we consider certain problems in hydromechanics and in the theory of anisotropic reinforced materials.

1. Formulation of the basic problem. Let ω_1 and ω_2 $(\operatorname{Im}\omega_1 = 0,\ \operatorname{Im}\omega_2 / \omega_1 > 0)$ be the fundamental periods of the piecewise-nonhomogeneous medium, dividing it into a set of congruent fundamental cells Π_{mn} (for example, into a set of periodic parallelograms). Since we assume the structure of all congruent cells to be identical, it is sufficient to describe the structure of cell Π_{00}. The basic structure of the cell Π_{00} consists of a group of distinct heterogeneous anisotropic inclusions D_j, bounded by the closed curves L_j $(j = 1, 2, \ldots, r)$. The nonuniformity of each of the domains D_j gives rise to a cell substructure, i. e. the presence in each of these domains of its own anisotropic inclusions d_{jq}, bounded by the closed curves l_{jq} $(j = 1, 2, \ldots, r;\ q = 1, 2, \ldots, r_j)$. We assume that the curves L_j and l_{jq} are simple smooth mutually disjunct Liapunov curves.

Let

$$L = \bigcup_{j=1}^{r} L_j, \quad d_j = \bigcup_{q=1}^{r_j} d_{jq}, \quad l_j = \bigcup_{q=1}^{r_j} l_{jq}, \quad B_j = D_j \setminus d_j$$

and let D be the unbounded domain occupaied by the basic homogeneous anisotropic